An Easy Expansion Exercise

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February 7, 1995

Exercise 3.5.11.(vii) of [Bar84], displays two lambda terms $(\lambda x.bx(bc))c$ and $(\lambda x.xx)(bc)$ due to Plotkin, having a common one step beta reduct $bc(bc)$, but which do not have a common beta expand.

Conjecture 3.2.38 of [Oos94], displays two other lambda terms $(\lambda x.a(b(x)))(c(d))$ and $a((\lambda y.b(c(y)))d)$, having a common one step beta reduct $a(b(c(d)))$, but which were conjectured not to have a common beta expand.

Here we prove this conjecture by elementary means.

Suppose $s \equiv (\lambda x.a(b(x)))(c(d)) \twoheadleftarrow r \twoheadrightarrow a((\lambda y.b(c(y)))d) \equiv t$. By standardisation [Bar84, p. 296 ff.] the common expand can be reached by standard expansions: $s \twoheadleftarrow_s r \twoheadrightarrow_s t$, which we may assume to differ already in the first step from r. Let us display the leftmost of these two redexes: $r \equiv \ldots (\lambda z.M)N \ldots$, and call the standard rewrite contracting this redex σ and the other one τ. By definition of standardisation, the displayed part $\ldots(\lambda z$ will be preserved by τ.

1. If τ ends in t, then since t contains only one λ, we must have that $z \equiv y$ and $r \equiv a((\lambda y.---$, which cannot rewrite to s,

2. If τ ends in s, then since s contains only one λ, we must have that $z \equiv x$ and $r \equiv (\lambda x.M)N$, with $M \twoheadrightarrow a(b(x))$ and $N \twoheadrightarrow c(d)$. Because of this last fact one can trace the descendants of N in the standard reduction $\sigma : r \equiv (\lambda x.M)N \rightarrow_s M[x := N] \twoheadrightarrow_s a((\lambda y.b(c(y)))d) \equiv t$. We know by confluence that any such descendant must reduce to $c(d)$. It it trivial to check that t does not contain subterms reducing to $c(d)$, so t does not contain descendants of N, hence we also have $M \equiv M[x := x] \twoheadrightarrow_s a((\lambda y.b(c(y)))d)$. Since $M \twoheadrightarrow a(b(x))$, this would contradict confluence.

The advantage of our terms over Plotkin's is that our terms are also elements of restricted lambda calculi such as linear and typed lambda calculus (the most complex type needed is $o \rightarrow o$). Moreover, the terms make clear that different ways of 'splitting' a term is one cause for failure of (local) confluence of beta expansion.

We conjecture that no term is confluent. In non-linear lambda calculi, this is easy to show via $s \leftarrow (\lambda x.s)(a(b(c(d))))$, but the linear case seems more difficult.

References

[Bar84] H. P. Barendregt. *The Lambda Calculus, Its Syntax and Semantics*, volume 103 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science Publishers B.V., Amsterdam, revised edition, 1984.

[Oos94] Vincent van Oostrom. *Confluence for Abstract and Higher-Order Rewriting*. PhD thesis, Vrije Universiteit, Amsterdam, March 1994.